Asymptotic Equation for Zeros of Hermite Polynomials from the Holstein-Primakoff Representation

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The Holstein-Primakoff representation for spin systems is used to derive expressions with solutions that are conjectured to be the zeros of Hermite polynomials $H_n(x)$ as $n \to \infty$. This establishes a correspondence between the zeros of the Hermite polynomials and the boundaries of the position basis of finite-dimensional Hilbert spaces.

The Hermite polynomials are prevalent in many fields. They can be defined as

$$H_n(x) = (-1)^n e^{x^n} \frac{d^n}{dx^n}\left(e^{-x^2}\right). \tag{1}$$

In the physics community, they are perhaps best recognized as the Gaussian-weighted eigenfunctions (in position representation) of the quantum harmonic oscillator (with $\hbar = m = \omega = 1$, a convention that will be used for the rest of the paper):

$$\frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right) e^{-\frac{x^2}{2}} H_n(x) = \left(n + \frac{1}{2}\right) e^{-\frac{x^2}{2}} H_n(x), \tag{2}$$

As such, they are orthogonal over the Gaussian-weighted whole domain, $\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{nm}$. This last property allows their use in Gaussian quadrature, a useful and popular numerical integration technique where $\int_{-\infty}^{\infty} f(x) dx$ is approximated as $\sum_{j=1}^{n} e^{-x_j^2} f(x_j)$ where x_j are the zeros of $H_n(x)$ and $f(x)$ is a well-behaved function. For this and many other reasons, an analytic formula for the asymptotic zeros of Hermite and other orthogonal polynomials has been a subject of much interest[1–7], especially in the applied mathematics community and the field of approximation theory.

In this paper, I examine the position state representation of the eigenstates of finite dimensional S-spin systems, as expressed in the Holstein-Primakoff transformation. As $S \to \infty$, the system becomes the infinite dimensional harmonic oscillator. This association allows me to derive the simple main results presented in eqs 6 and 7, with solutions that I conjecture become the asymptotic zeros of the Hermite polynomials (as $n \to \infty$). Furthermore, I numerically show that this convergence is rather quick and so the expressions can frequently be used, in many instances of finite-precision application, as the effective zeros of $H_n(x)$ with finite n, such as in applications of Gaussian quadrature. In a more aesthetic sense, these results also establish a beautiful correspondence between the boundaries of equal area partitions of circles with radii that are increasing in a certain manner and the Hermite polynomial zeros.

Spin systems are defined by the fundamental commutation relations between operators $\hat{S}^z$, $\hat{S}^+$ and $\hat{S}^-$:

$$\left[\hat{S}^z, \hat{S}^+\right] = \hat{S}^+, \ \left[\hat{S}^z, \hat{S}^-\right] = \hat{S}^-, \ \left[\hat{S}^+, \hat{S}^-\right] = 2\hat{S}^z. \tag{3}$$

Associating a spin with a boson $c^\dagger$, Holstein and Primakoff showed that to satisfy these commutation relations, the operators can be expressed as[8]

$$\hat{S}^z = \hat{c}^\dagger \hat{c} - S, \tag{4}$$

$$\hat{S}^+ = \hat{c}^\dagger \sqrt{2S - \hat{c}^\dagger \hat{c}}, \ \text{and} \ \hat{S}^- = \sqrt{2S - \hat{c}^\dagger \hat{c}}\, \hat{c}. \tag{5}$$

This is a very useful association and has found many applications in the condensed matter field's study of many-body spin systems. Each boson excitation represents the "ladder up" finitesimal excitation away from the spin's extremal S state. The Hilbert space is finite-dimensional and possesses $2S + 1$ states $\{-S, -S + 1, \ldots, S\}$. In fact, considering eq. 5 it is clear that the Hilbert space outside this defined space is not even Hermitian.

Transforming from the Holstein-Primakoff bosonic representation to position (and its conjugate momentum) space (using the relations $c^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$ and $c = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$) reveals that the trivial Hamiltonian is the harmonic oscillator: $\hat{H} = \hat{S}_z = \frac{1}{2}\left(\hat{q}^2 + \hat{p}^2\right) - \left(S + \frac{1}{2}\right)$. Moreover, transformation of the $\hat{S}^+$ and $\hat{S}^-$ in eq. 5 reveals that the Hilbert space spans the domain $r^2 \equiv p^2 + q^2 \leq \sqrt{4S + 1}$. Just as in the S_z representation, $2S$ states all with the same area must exist within this domain. Fig. 1 sketches out what they look like for the $\{S = \frac{1}{2}, S = 1, S = \frac{3}{2}\}$-spin systems.

For a particular S-spin system, the lowest eigenstate must have the same sign at all q-basis elements since it must be nodeless. On the other hand, the highest eigenstate must have $n - 1$ nodes and so the q-basis elements must alternate in sign such that the eigenfunction passes through zero between them. This latter behavior is sketched in fig. 1 in red by the Hermite polynomial $H_n(x)$ denoting the value of the overlying q-basis element for the highest eigenstate.

For $S \to \infty$, the Hilbert space becomes infinite-dimensional and the Hamiltonian becomes that of the

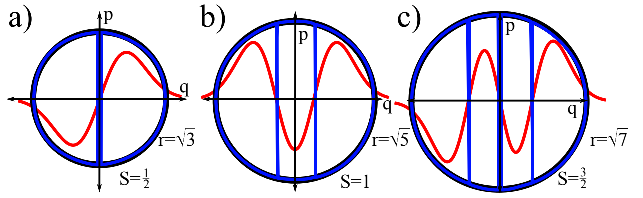

FIG. 1. The q-basis representation of a) $S = \frac{1}{2}$, b) $S = 1$ and c) $S = \frac{3}{2}$ systems is shown. The radius of the Hilbert space's domain is equal to $\sqrt{4S+1}$ and so grows along with the number of allowed basis elements.

harmonic oscillator defined over $(p, q) \in \mathbb{R}^2$ with the associated eigenfunctions proportional to $e^{-\frac{x^2}{2}} H_n(x)$. It therefore follows that as $S \to \infty$, the boundaries of the q-basis elements become the zeros of the Hermite polynomial $H_n(x)$ where $n = 2S$ since the highest eigenstate must still have alternating sign with each q-basis element.

Hermite polynomial zeros x_j are real and symmetric around $x = 0$. To determine these boundary points, the $2S$-dimensional Hilbert space's circular shape in position space can be exploited. For even $2S$, the area of the all the q-basis elements up until the jth boundary (measuring from the origin) is $\pi r^2 \frac{2j-1}{n+1}$. For odd $2S$, the area is $\pi r^2 \frac{2j}{n+1}$. This is illustrated in fig. 2.

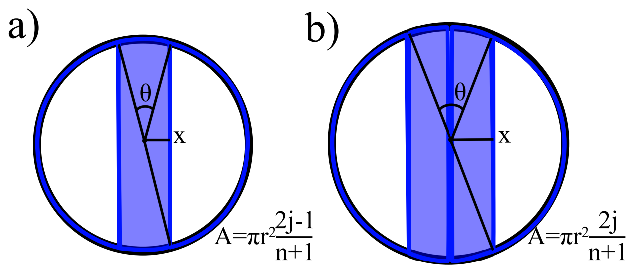

FIG. 2. The area of the central a) $2j - 1$ or b) $2j$ q-basis elements that approximately determine the jth zero of the Hermite polynomial $H_n(x)$ for n even and odd respectively is shaded in blue. The approximate jth zero is at the right boundary of these regions.

Using simple relations for the area of circle sectors and rectangles, it is possible to relate these q-basis element areas to x_j; The equation involving the approximate zeros of Hermite polynomials H_n with n even is:

$$\frac{(2j-1)\pi}{n+1} = \sin\theta + \theta, \tag{6}$$

while for odd n it is:

$$\frac{2j\pi}{n+1} = \sin\theta + \theta, \tag{7}$$

where $\theta = 2\sin^{-1}\frac{x_j}{r}$ and $r = \sqrt{2n+1}$.

Solving these equations for x_j yields the approximate jth zero for the nth Hermite polynomial. The results for the zeros of the first 50 Hermite polynomials are compared to the exact zeros in fig. 3. In both cases, eqs. 6 and 7 converge to the zeros of the Hermite functions quite quickly[9].

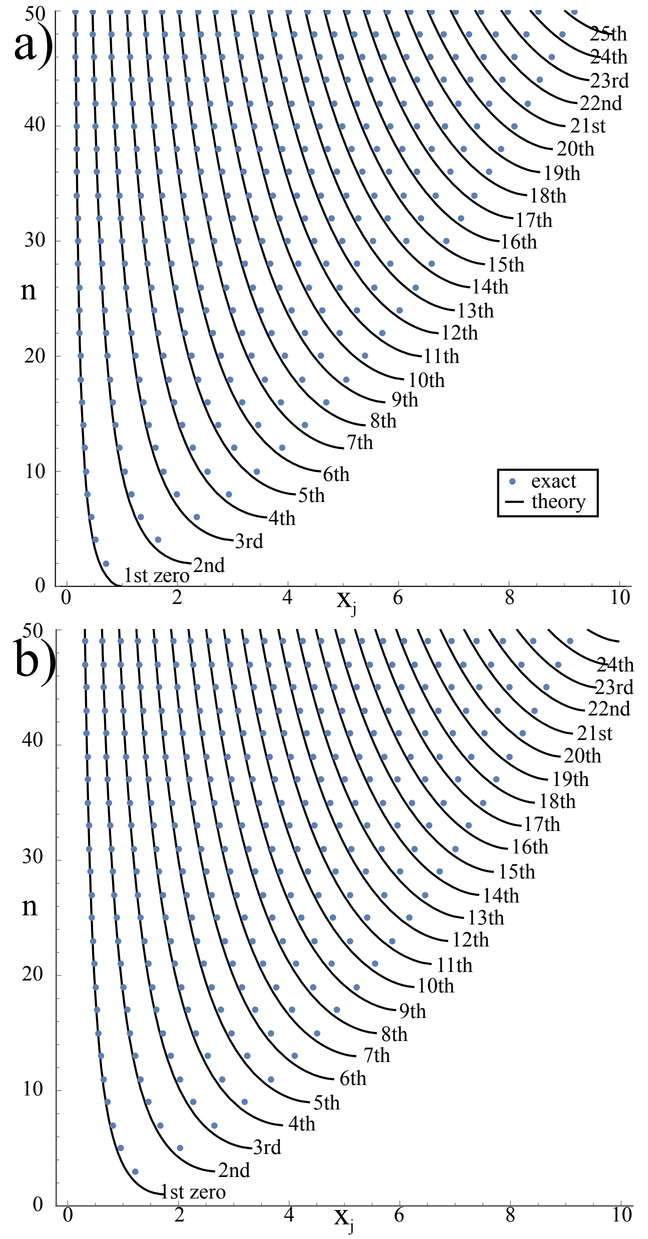

FIG. 3. Exact jth zeros of the Hermite polynomials $H_n(x)$ for n a) even and b) odd compared to those obtained from solving eqs. 6 and 7.

The finding that the boundaries of equal area partitions of growing circles correspond to the asymptotic zeros of the Hermite functions appears to be a novel one from a search of the literature. It is all the more surprising that the origin of this one-to-one correspondance stems from the Holstein-Primakoff representations for finite-dimensional spin systems. Furthermore, on a practical level, the apparently rapid convergence of these

solutions suggests that they may be useful for more efficient determination of Hermite polynomial zeros for large-dimensional implementations of Gaussian quadrature.

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[9] J. Katriel, through correspondence, showed that eqns 6 and 7 agree with the first asymptotic term from Dominici[6] for $n \to \infty$ for low j (not for maximal j). The latter result makes sense from the point of view that the maximal x_j is always close to the edge of the Hilbert space where the wavefunction goes to zero for any finite n whereas that of the harmonic oscillator decays forever. Eqs. 6 and 7 do not agree with higher order terms (w.r.t. $\frac{1}{n}$) in Dominici's asymptotic expansion.